BİTLİS EREN ÜNİVERSİTESİ
FEN BİLİMLERİ DERGİSİ

ISSN: 2147-3129 / e-ISSN: 2147-3188

Article Type	: Research Article	**Year**	: 2025
Received	: July 4, 2025	**Volume**	: 14
Revised	: December 10, 2025	**Issue**	: 4
Accepted	: December 12, 2025	**Pages**	: 2357-2373
DOI	: 10.17798/bitlisfen.1735128		

ON THE BICUBIC BÉZIER SURFACES AND PARABOLOIDS IN E^3

Şeyda KILIÇOĞLU [1], **Semra YURTTANÇIKMAZ** [2,*]

[1] *Başkent University, Faculty of Education, Department of Mathematics, Ankara, Türkiye*

[2] *Atatürk University, Faculty of Science, Department of Mathematics, Erzurum, Türkiye*

**** Corresponding Author:*** *semrakaya@atauni.edu.tr*

ABSTRACT

Bicubic Bézier surfaces are fundamental in the world of computer graphics, computer-aided design (CAD), and animation because they offer a powerful balance of flexibility, smoothness, and control. In addition to being compatible with other surface representations, they have attracted the attention of scientists due to their mathematical strength and understandability, and many studies have been conducted on this subject.

In this paper, first we examine the matrix representation of bicubic Bézier surfaces whose control points lie in E^3. Second, as examples, we consider elliptic and hyperbolic paraboloids as bicubic Bézier surfaces. Finally, we present a method for determining the control points of a given elliptic paraboloid, hyperbolic paraboloid, and parabolic cylinder as bicubic Bézier surfaces.

Keywords: Bicubic Bézier surfaces, Control points, Matrix representation, Elliptic paraboloid, Hyperbolic paraboloid, Parabolic cylinder.

1 INTRODUCTION

Bézier surfaces are very important because they produce smooth and curved surfaces, offer flexible and intuitive modeling with control points, provide digital representations of real-world objects such as vehicle bodies, architectural structures, character models, and are the basic components of CAD and graphics software [1]-[2]-[3]-[22]-[24]. Bicubic Bézier surfaces are a special type of Bézier surface. A bicubic Bézier surface is a type of smooth, curved surface defined mathematically using Bézier curves of degree 3 in two directions (u and v). It is

License: CC BY-NC 4.0

determined by 16 control points arranged in a 4×4 grid. Bicubic Bézier surfaces are widely used because they can model smooth, flexible surfaces in a mathematically precise and visually appealing way [4]-[5]-[6]-[26]. Bicubic Bézier surfaces have a very important place because they provide a high level of flexibility and smoothness, are ideal for realistic surface modeling such as car bodies, character surfaces, terrains, are widely used in areas such as scientific visualization, industrial design and film/game graphics, can be combined with other surfaces without seams and therefore facilitate the creation of large and complex models [23]-[25]. Bézier surfaces are derived directly from Bézier curves in terms of control points, Bernstein polynomials and parameterization. For this reason, the theory of curves is frequently used when conducting Bézier surface research. Bézier curves are mathematical curves that are widely used especially in computer graphics, animation, vector drawing programs (such as Adobe Illustrator, CorelDRAW), CAD systems and font designs. They were developed by French engineer Pierre Bézier in automotive design. A Bézier curve is a parametric curve defined using a series of control points. The curve is "driven" by these points, meaning that the shape of the curve depends on the location of these control points. Its degree is determined by the number of control points that define the curve [7]-[8]-[9]-[10]-[11]. Among Bézier curves of different degrees, third degree Bézier curves are the most widely used. This is because, with 4 control points, complex and fluid shapes can be easily defined, and since its mathematical formula is easy and understandable, it offers a good balance between complexity and control [12]-[13]-[14]-[15]. While control points define the geometry of Bézier curves and surfaces and provide intuitive control, matrix representations offer ease of calculation, transformation operations and hardware compatibility [16].

In this article, matrix representations of bicubic Bézier surfaces with control points in 3D Euclidean space are presented. Investigating bicubic Bézier surfaces is vital because they form a core building block of surface modeling. Improving our understanding of them drives progress across graphics, design, manufacturing, and computational geometry. The literature values them for both their practical applications and their theoretical richness. For this reason, the control points of bicubic Bézier surfaces of some special surface types such as elliptic paraboloid, hyperbolic paraboloid, parabolic cylinder are determined and the matrix representations of these surfaces are calculated.

2 MATERIAL AND METHOD

A Bézier curve is a parametric curve defined by a set of control points. These curves are widely used in computer graphics, animation, and design because they are simple to compute and offer intuitive control over the shape of the curve. A Bézier curve of degree n (also called an n-degree Bézier curve) is defined by $n+1$ control points $P_0, P_1, P_2, \dots, P_n$ and is given by the following parametric equation:

$$B(t) = \sum_{i=0}^{n} P_i B_{i,n}(t) \tag{1}$$

where:

- $B(t)$ is the Bézier curve at parameter value t (where t ranges from 0 to 1).
- P_i are the control points.
- $B_{i,n}(t)$ are the Bernstein basis polynomials, which are defined as:

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i} \tag{2}$$

$\binom{n}{i}$ is the binomial coefficient [1]-[2]-[17].

Definition 2.1: A cubic Bézier curve, which is a special Bézier curve given by four control points P_0, P_1, P_2 and P_3, is expressed with the following parameterization

$$\alpha(t) = (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t) P_2 + t^3 P_3 \tag{3}$$

and its matrix form is

$$\alpha(t) = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T [B^3] \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} \tag{4}$$

where $[B^3] = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. For more details, see [17].

Bézier surfaces are a generalization of the Bézier curve into three dimensions. They are used to define smooth, curved surfaces in 3D space, and are often employed in 3D modeling and computer-aided design (CAD). A Bézier surface is defined by a **grid of control points.** For the surface defined by $m+1$ control points along the u-direction and $n+1$ control points along the v-direction, the surface $B(u,v)$ is given by the following parametric equation:

$$B(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} P_{i,j} B_{i,m}(u) B_{j,n}(v) \tag{5}$$

where:

- $B(u,v)$ is the Bézier surface at parameters u and v, where $u, v \in [0,1]$.
- $P_{i,j}$ are the control points arranged in a grid.
- $B_{i,m}(u)$ and $B_{j,n}(v)$ are the Bernstein basis polynomials for u and v, respectively, and are defined as:

$$B_{i,m}(u) = \binom{m}{i} u^i (1-u)^{m-i}, \qquad B_{j,n}(v) = \binom{n}{j} v^j (1-v)^{n-j}. \tag{6}$$

Definition 2.2: Consider a Bézier surface of degree (n,m) given by set of $P_{i,j}$ of control points $(n+1)(m+1)$, where $i = 0,\dots,n$ and $j = 0,\dots,m$. Parametric expression of two-dimensional Bézier surface with respect to parameters u, v can be introduced as follows

$$\boldsymbol{B}(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} [P_{i,j}] \binom{n}{i} u^i (1-u)^{n-i} \binom{m}{j} v^j (1-v)^{n-j}$$
$$\boldsymbol{B}(u,v) = [v] M_v P M_u^T [u]^T. \tag{7}$$

For more details, see [18].

In space, $u = constant$, $v = constant$ lines, also known as coordinate lines, as well as the edges of a deformed (u,v) unit square, can be given as examples of Bézier curves. Because a Bézier surface lies within a hull formed by control points, it will remain within the bounding box formed by these control points in any given Cartesian coordinate system. Although a Bézier surface generally does not pass through its control points, the corners of the deformed unit square coincide with four of the control points and this surface transforms in the same way as its control points under all linear transformations. In [21], the implicitization of parametric polynomial surfaces has been proposed.

3 RESULTS AND DISCUSSION

3.1 Bicubic Bézier Surface in E^3

Bézier surfaces are most frequently utilized as collections of bicubic patches, where both degrees in the parametric directions are set to 3. Each individual bicubic patch is fully determined by 16 control points arranged in a 4×4 grid. These patches are often connected

together to construct a B-spline surface, in much the same way that individual Bézier curve segments are combined to create a B-spline curve.

Definition 3.1. For $n = m = 3$, a given bicubic Bézier surface of degree (3,3) is defined by a set of $(3+1)(3+1) = 16$ control points $k_{i,j}$ where $i = 0,\ldots,3$ and $j = 0,\ldots,3$

$$B(u,v) = \sum_{i,j=0}^{3} \binom{3}{i} u^i (1-u)^{n-i} \binom{3}{j} v^j (1-v)^{n-j} [k_{i,j}]$$

$$B(u,v) = [v][B^3][P_{ij}][B^3][u]^T \tag{8}$$

$$B(u,v) = [v]\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} P_{00} & P_{10} & P_{20} & P_{30} \\ P_{01} & P_{11} & P_{21} & P_{31} \\ P_{02} & P_{12} & P_{22} & P_{32} \\ P_{03} & P_{13} & P_{23} & P_{33} \end{bmatrix}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}[u]^T$$

where $[P_{ij}]$ the control points bicubic Bézier surface. For more detail see [18]-[19]. Also any bicubic Bézier surface can be written by the following matrix representation or bicubic Bézier surface has the following matrix form with $[v] = [v^3 \quad v^2 \quad v \quad 1]$ and $[u] = [u^3 \quad u^2 \quad u \quad 1]$

$$B(u,v) = \begin{bmatrix} [v][B^3][P_x][B^3][u]^T \\ [v][B^3][P_y][B^3][u]^T \\ [v][B^3][P_z][B^3][u]^T \end{bmatrix} \tag{9}$$

where $[P_x], [P_y], [P_z]$ are the components of the control points for the bicubic Bézier surface and and are written as follows

$$[P_x] = \begin{bmatrix} P_{0x} & P_{4x} & P_{8x} & P_{12x} \\ P_{1x} & P_{5x} & P_{9x} & P_{13x} \\ P_{2x} & P_{6x} & P_{10x} & P_{14x} \\ P_{3x} & P_{7x} & P_{11x} & P_{15x} \end{bmatrix}, \quad [P_y] = \begin{bmatrix} P_{0y} & P_{4y} & P_{8y} & P_{12y} \\ P_{1y} & P_{5y} & P_{9y} & P_{13y} \\ P_{2y} & P_{6y} & P_{10y} & P_{14y} \\ P_{3y} & P_{7y} & P_{11y} & P_{15y} \end{bmatrix}, \quad [P_z] = \begin{bmatrix} P_{0z} & P_{4z} & P_{8z} & P_{12z} \\ P_{1z} & P_{5z} & P_{9z} & P_{13z} \\ P_{2z} & P_{6z} & P_{10z} & P_{14z} \\ P_{3z} & P_{7z} & P_{11z} & P_{15z} \end{bmatrix}. \tag{10}$$

For more details see [20].

Theorem 3.1. The x, y and z components of the control points for the any parametric surfaces as cubic Bézier surface are

$$\begin{aligned} [P_x] &= [B^3]^{-1}[CMPS]_x[B^3]^{-1} \\ [P_y] &= [B^3]^{-1}[CMPS]_y[B^3]^{-1} \\ [P_z] &= [B^3]^{-1}[CMPS]_z[B^3]^{-1} \end{aligned} \tag{11}$$

where $[CMPS]_x, [CMPS]_y, [CMPS]_z$ are coefficients matrices of parametric surfaces for component x, y, z, respectively.

Proof. The parametric form of any surface $\varphi(u,v) = (u, v, F(u,v))$ can be written with the following matrix representation with $[u]$ and $[v]$

$$\varphi(u,v) = \left([v][CMPS]_x[u]^T, [v][CMPS]_y[u]^T, [v][CMPS]_z[u]^T\right) \tag{12}$$

Also surface $\varphi(u,v)$, as bicubic Bézier surface it has the following matrix form

$$\varphi(u,v) = \begin{bmatrix} [v][B^3][P_x][B^3][u]^T \\ [v][B^3][P_y][B^3][u]^T \\ [v][B^3][P_z][B^3][u]^T \end{bmatrix} \tag{13}$$

Using the equality of (3.2) and (3.3) we get

$$\begin{aligned} [v][CMPS]_x[u]^T &= [v][B^3][P_x][B^3][u]^T \\ [v][CMPS]_y[u]^T &= [v][B^3][P_y][B^3][u]^T \\ [v][CMPS]_z[u]^T &= [v][B^3][P_z][B^3][u]^T \end{aligned} \tag{14}$$

If we solve these matrix equations we have

$$\begin{aligned} [B^3]^{-1}[CMPS]_x[B^3]^{-1} &= [P_x] \\ [B^3]^{-1}[CMPS]_y[B^3]^{-1} &= [P_y] \\ [B^3]^{-1}[CMPS]_z[B^3]^{-1} &= [P_z] \end{aligned} \tag{15}$$

This completes the proof.

3.2 Paraboloids as Bicubic Bézier Surface

In mathematics, a paraboloid is a type of quadric surface characterized by having a single axis of symmetry and lacking a center of symmetry. The name originates from the word *parabola*, a conic section that also exhibits this kind of symmetrical behaviour. When a plane cuts through a paraboloid parallel to its axis of symmetry, the resulting intersection is always a parabola.

Paraboloids come in two main forms: elliptic and hyperbolic. A hyperbolic paraboloid is defined by the fact that other planar cross-sections (not parallel to the axis) produce either hyperbolas or, in the case of a tangent plane, intersecting straight lines. An elliptic paraboloid, on the other hand, yields ellipses or, when intersected by a tangent plane, a single point. All paraboloids, whether elliptic or hyperbolic, can be described as translation surfaces they can be generated by moving one parabola along the path defined by another parabola. Alternatively, a paraboloid can be described as a non-cylindrical quadric surface whose quadratic part in its implicit equation can be factored over the complex numbers into two distinct linear terms. If these factors are real, the paraboloid is hyperbolic; if they are complex conjugates, it is elliptic.

3.2.1 Oriented Elliptic Paraboloid

An **elliptic paraboloid** resembles the shape of a rounded bowl or cup. When its axis is aligned vertically, the surface has either a **minimum or maximum point,** depending on the direction it opens. In an appropriately chosen coordinate system with axes labeled x, y, and z the surface can be described by the equation:

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \tag{16}$$

Here, the constants a and b determine the degree of curvature in the xz **−plane** and yz **−plane,** respectively. When written in this form, the paraboloid opens **upward** along the z -axis. The parametric form of an eliptic paraboloid is $\varphi(u,v) = \left(u, v, \frac{u^2}{a^2} + \frac{v^2}{b^2}\right)$.

Theorem 3.2.1. The control points of the oriented elliptic paraboloid surface $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ as bicubic Bézier surface are

$$\begin{matrix}[P_{00} & P_{01} & P_{02} & P_{03} & P_{10} & P_{11} & P_{12} & P_{13} & P_{20} & P_{21} & P_{22} & P_{23} & P_{30} & P_{31} & P_{32} & P_{33}]\end{matrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 1/3 & 2/3 & 2/3 & 2/3 & 2/3 & 1 & 1 & 1 & 1 \\ 0 & 1/3 & 2/3 & 1 & 0 & 1/3 & 2/3 & 1 & 0 & 1/3 & 2/3 & 1 & 0 & 1/3 & 2/3 & 1 \\ 0 & 0 & \frac{1}{3b^2} & \frac{1}{b^2} & 0 & 0 & \frac{1}{3b^2} & \frac{1}{b^2} & \frac{1}{3a^2} & \frac{1}{3a^2} & \frac{1}{3a^2}+\frac{1}{3b^2} & \frac{1}{3a^2}+\frac{1}{b^2} & \frac{1}{a^2} & \frac{1}{a^2} & \frac{1}{a^2}+\frac{1}{3b^2} & \frac{1}{a^2}+\frac{1}{b^2} \end{bmatrix}. \tag{17}$$

Proof. We can write the oriented elliptic paraboloid surface as in the parametric form $\varphi(u,v) = \left(u, v, \frac{u^2}{a^2} + \frac{v^2}{b^2}\right)$. Also the matrix representation is

$$\varphi(u,v) = \left([v] \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} [u]^T, [v] \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} [u]^T, [v] \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{b^2} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{a^2} & 0 & 0 \end{bmatrix} [u]^T \right) \tag{18}$$

hence we have

$$[CMPS]_x = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, [CMPS]_y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, [CMPS]_z = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{b^2} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{a^2} & 0 & 0 \end{bmatrix}. \tag{19}$$

By using theorem 3.1, we get

$$[P_x] = [B^3]^{-1}[CMPS]_x[B^3]^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix} \tag{20}$$

$$[P_y] = [B^3]^{-1}[CMPS]_y[B^3]^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{21}$$

$$[P_z] = [B^3]^{-1}[CMPS]_z[B^3]^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{b^2} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{a^2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{3a^2} & \frac{1}{a^2} \\ 0 & 0 & \frac{1}{3a^2} & \frac{1}{a^2} \\ \frac{1}{3b^2} & \frac{1}{3b^2} & \frac{1}{3a^2}+\frac{1}{3b^2} & \frac{1}{a^2}+\frac{1}{3b^2} \\ \frac{1}{b^2} & \frac{1}{b^2} & \frac{1}{3a^2}+\frac{1}{b^2} & \frac{1}{a^2}+\frac{1}{b^2} \end{bmatrix} \tag{22}$$

and we have the coordinates the control points as in the following way

$$\begin{bmatrix} P_{0x} & P_{4x} & P_{8x} & P_{12x} \\ P_{1x} & P_{5x} & P_{9x} & P_{13x} \\ P_{2x} & P_{6x} & P_{10x} & P_{14x} \\ P_{3x} & P_{7x} & P_{11x} & P_{15x} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix}, \quad \begin{bmatrix} P_{0y} & P_{4y} & P_{8y} & P_{12y} \\ P_{1y} & P_{5y} & P_{9y} & P_{13y} \\ P_{2y} & P_{6y} & P_{10y} & P_{14y} \\ P_{3y} & P_{7y} & P_{11y} & P_{15y} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ 1 & 1 & 1 & 1 \end{bmatrix}, \tag{23}$$

$$\begin{bmatrix} P_{0z} & P_{4z} & P_{8z} & P_{12z} \\ P_{1z} & P_{5z} & P_{9z} & P_{13z} \\ P_{2z} & P_{6z} & P_{10z} & P_{14z} \\ P_{3z} & P_{7z} & P_{11z} & P_{15z} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{3a^2} & \frac{1}{a^2} \\ 0 & 0 & \frac{1}{3a^2} & \frac{1}{a^2} \\ \frac{1}{3b^2} & \frac{1}{3b^2} & \frac{1}{3a^2}+\frac{1}{3b^2} & \frac{1}{a^2}+\frac{1}{3b^2} \\ \frac{1}{b^2} & \frac{1}{b^2} & \frac{1}{3a^2}+\frac{1}{b^2} & \frac{1}{a^2}+\frac{1}{b^2} \end{bmatrix}. \tag{24}$$

Theorem 3.2.2. The control points of the oriented circular paraboloid $z = \frac{x^2}{a^2} + \frac{y^2}{a^2}$ as bicubic Bézier surface are

$$\begin{matrix} [P_{00} & P_{01} & P_{02} & P_{03} & P_{10} & P_{11} & P_{12} & P_{13} & P_{20} & P_{21} & P_{22} & P_{23} & P_{30} & P_{31} & P_{32} & P_{33}] \end{matrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 1/3 & 2/3 & 2/3 & 2/3 & 2/3 & 1 & 1 & 1 & 1 \\ 0 & 1/3 & 2/3 & 1 & 0 & 1/3 & 2/3 & 1 & 0 & 1/3 & 2/3 & 1 & 0 & 1/3 & 2/3 & 1 \\ 0 & 0 & \frac{1}{3a^2} & \frac{1}{a^2} & 0 & 0 & \frac{1}{3a^2} & \frac{1}{a^2} & \frac{1}{3a^2} & \frac{1}{3a^2} & \frac{2}{3a^2} & \frac{4}{3a^2} & \frac{1}{a^2} & \frac{1}{a^2} & \frac{4}{3a^2} & \frac{2}{a^2} \end{bmatrix}. \tag{25}$$

Proof: It is trivial for $a = b$.

Theorem 3.2.3. The bicubic Bézier surface, whose components of control points are given as follows

$$\begin{bmatrix} P_{0x} & P_{0y} & P_{0z} \\ P_{1x} & P_{1y} & P_{1z} \\ P_{2x} & P_{2y} & P_{2z} \\ P_{3x} & P_{3y} & P_{3z} \\ P_{4x} & P_{4y} & P_{4z} \\ P_{5x} & P_{5y} & P_{5z} \\ P_{6x} & P_{6y} & P_{6z} \\ P_{7x} & P_{7y} & P_{7z} \\ P_{8x} & P_{8y} & P_{8z} \\ P_{9x} & P_{9y} & P_{9z} \\ P_{10x} & P_{10y} & P_{10z} \\ P_{11x} & P_{11y} & P_{11z} \\ P_{12x} & P_{12y} & P_{12z} \\ P_{13x} & P_{13y} & P_{13z} \\ P_{14x} & P_{14y} & P_{14z} \\ P_{15x} & P_{15y} & P_{15z} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3b^2} & \frac{2}{3} \\ 0 & \frac{1}{b^2} & 1 \\ \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3b^2} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{b^2} & 1 \\ \frac{2}{3} & \frac{1}{3a^2} & 0 \\ \frac{2}{3} & \frac{1}{3a^2} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3a^2}+\frac{1}{3b^2} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3a^2}+\frac{1}{b^2} & 1 \\ 1 & \frac{1}{a^2} & 0 \\ 1 & \frac{1}{a^2} & \frac{1}{3} \\ 1 & \frac{1}{a^2}+\frac{1}{3b^2} & \frac{2}{3} \\ 1 & \frac{1}{a^2}+\frac{1}{b^2} & 1 \end{bmatrix} \tag{26}$$

is an elliptic paraboloid with the parametric equation $\varphi(u,v) = \left(u, \frac{u^2}{a^2} + \frac{v^2}{b^2}, v\right)$.

Proof: If the given control points are substituted in, taking Theorem 3.1 into account, the parametric equation of elliptic paraboloid is obtained as follows

$$\varphi(u,v) = \begin{pmatrix} [v]\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}[u]^T, \\ [v]\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & \frac{1}{3a^2} & \frac{1}{a^2} \\ 0 & 0 & \frac{1}{3a^2} & \frac{1}{a^2} \\ \frac{1}{3b^2} & \frac{1}{3b^2} & \frac{1}{3a^2}+\frac{1}{3b^2} & \frac{1}{a^2}+\frac{1}{3b^2} \\ \frac{1}{b^2} & \frac{1}{b^2} & \frac{1}{3a^2}+\frac{1}{b^2} & \frac{1}{a^2}+\frac{1}{b^2} \end{bmatrix}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}[u]^T, \\ [v]\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}[u]^T \end{pmatrix} \tag{27}$$

$$\varphi(u,v) = \left(u, \frac{1}{a^2b^2}(a^2v^2 + b^2u^2), v\right) = \left(u, \frac{u^2}{a^2} + \frac{v^2}{b^2}, v\right) \tag{28}$$

This completes the proof.

Example 3.2.1. Lets show that the bicubic Bézier surface with given 16 control points as follows

$$\begin{aligned} &P_{00}=(0,0,0), \quad P_{10}=\left(\frac{1}{3},0,0\right), \quad P_{20}=\left(\frac{2}{3},0,\frac{1}{3}\right), \quad P_{30}=(1,0,1),\\ &P_{01}=\left(0,\frac{1}{3},0\right), \quad P_{11}=\left(\frac{1}{3},\frac{1}{3},0\right), \quad P_{21}=\left(\frac{2}{3},\frac{1}{3},\frac{1}{3}\right), \quad P_{31}=\left(1,\frac{1}{3},1\right),\\ &P_{02}=\left(0,\frac{2}{3},\frac{1}{3}\right), \quad P_{12}=\left(\frac{1}{3},\frac{2}{3},\frac{1}{3}\right), \quad P_{22}=\left(\frac{2}{3},\frac{2}{3},\frac{2}{3}\right), \quad P_{32}=\left(1,\frac{2}{3},\frac{4}{3}\right),\\ &P_{03}=(0,1,1), \quad P_{13}=\left(\frac{1}{3},1,1\right), \quad P_{23}=\left(\frac{2}{3},1,\frac{4}{3}\right), \quad P_{33}=(1,1,2). \end{aligned} \tag{29}$$

is an oriented circular paraboloid surface.

Proof. If we use the following equation

$$\begin{aligned} \varphi(u,v) &= [v]\begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 3 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}P_{00} & P_{10} & P_{20} & P_{30}\\ P_{01} & P_{11} & P_{21} & P_{31}\\ P_{02} & P_{12} & P_{22} & P_{32}\\ P_{03} & P_{13} & P_{23} & P_{33}\end{bmatrix}\begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 3 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}[u]^T \\ &= [-(v-1)^3 \quad 3v(v-1)^2 \quad -3v^2(v-1) \quad v^3]\begin{bmatrix}P_{00} & P_{10} & P_{20} & P_{30}\\ P_{01} & P_{11} & P_{21} & P_{31}\\ P_{02} & P_{12} & P_{22} & P_{32}\\ P_{03} & P_{13} & P_{23} & P_{33}\end{bmatrix}\begin{bmatrix}-(u-1)^3\\ 3u(u-1)^2\\ -3u^2(u-1)\\ u^3\end{bmatrix} \end{aligned} \tag{30}$$

by simplifying we get the following equation

$$\begin{aligned} \varphi(u,v) = (&-u(u^2-1), \quad -3uv(u-1), \quad -u(3v^2+u)(u-1))\\ &+(u^3, \quad v(3u^2-3u+1), \quad u^3+3u^2v^2-3uv^2+v^2) = (u, \quad v, \quad u^2+v^2) \end{aligned} \tag{31}$$

which is the parametric form of the oriented circular paraboloid $z=x^2+y^2$.

Example 3.2.2. Lets show that the bicubic Bézier surface with the given components of control points as follows

$$[P_x]=\begin{bmatrix}0 & \frac{1}{3} & \frac{2}{3} & 1\\ 0 & \frac{1}{3} & \frac{2}{3} & 1\\ 0 & \frac{1}{3} & \frac{2}{3} & 1\\ 0 & \frac{1}{3} & \frac{2}{3} & 1\end{bmatrix}, [P_y]=\begin{bmatrix}0 & 0 & 0 & 0\\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3}\\ 1 & 1 & 1 & 1\end{bmatrix}, [P_z]=\begin{bmatrix}0 & 0 & \frac{1}{75} & \frac{1}{25}\\ 0 & 0 & \frac{1}{75} & \frac{1}{25}\\ \frac{1}{27} & \frac{1}{27} & \frac{34}{675} & \frac{52}{675}\\ \frac{1}{9} & \frac{1}{9} & \frac{28}{225} & \frac{34}{225}\end{bmatrix} \tag{32}$$

is an oriented elliptic paraboloid $z=\frac{x^2}{25}+\frac{y^2}{9}$.

Proof. If we use the theorem 3.1 as in the following way, we get

$$\varphi(u,v)=\begin{pmatrix}\begin{bmatrix}v^3\\v^2\\v\\1\end{bmatrix}^T\begin{bmatrix}-1&3&-3&1\\3&-6&3&0\\-3&3&0&0\\1&0&0&0\end{bmatrix}\begin{bmatrix}0&\frac{1}{3}&\frac{2}{3}&1\\0&\frac{1}{3}&\frac{2}{3}&1\\0&\frac{1}{3}&\frac{2}{3}&1\\0&\frac{1}{3}&\frac{2}{3}&1\end{bmatrix}\begin{bmatrix}-1&3&-3&1\\3&-6&3&0\\-3&3&0&0\\1&0&0&0\end{bmatrix}\begin{bmatrix}u^3\\u^2\\u\\1\end{bmatrix},\\ \begin{bmatrix}v^3\\v^2\\v\\1\end{bmatrix}^T\begin{bmatrix}-1&3&-3&1\\3&-6&3&0\\-3&3&0&0\\1&0&0&0\end{bmatrix}\begin{bmatrix}0&0&0&0\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\\\frac{2}{3}&\frac{2}{3}&\frac{2}{3}&\frac{2}{3}\\1&1&1&1\end{bmatrix}\begin{bmatrix}-1&3&-3&1\\3&-6&3&0\\-3&3&0&0\\1&0&0&0\end{bmatrix}\begin{bmatrix}u^3\\u^2\\u\\1\end{bmatrix},\\ \begin{bmatrix}v^3\\v^2\\v\\1\end{bmatrix}^T\begin{bmatrix}-1&3&-3&1\\3&-6&3&0\\-3&3&0&0\\1&0&0&0\end{bmatrix}\begin{bmatrix}0&0&\frac{1}{75}&\frac{1}{25}\\0&0&\frac{1}{75}&\frac{1}{25}\\\frac{1}{27}&\frac{1}{27}&\frac{34}{675}&\frac{52}{675}\\\frac{1}{9}&\frac{1}{9}&\frac{28}{225}&\frac{34}{225}\end{bmatrix}\begin{bmatrix}-1&3&-3&1\\3&-6&3&0\\-3&3&0&0\\1&0&0&0\end{bmatrix}\begin{bmatrix}u^3\\u^2\\u\\1\end{bmatrix}\end{pmatrix}=\left(u,v,\frac{1}{25}u^2+\frac{1}{9}v^2\right) \quad (33)$$

Example 3.2.3. Lets find the control points of the elliptic paraboloid $y=\frac{x^2}{25}+\frac{z^2}{9}$ as bicubic Bézier surface. First we can write the parametric form of the elliptic paraboloid as $\varphi(u,v)=\left(u,\frac{u^2}{25}+\frac{v^2}{9},v\right)$. By replacing $a=5$ and $b=3$ in theorem 3.2.1, the control points are

$$[P_{00}\ P_{01}\ P_{02}\ P_{03}\ P_{10}\ P_{11}\ P_{12}\ P_{13}\ P_{20}\ P_{21}\ P_{22}\ P_{23}\ P_{30}\ P_{31}\ P_{32}\ P_{33}]$$
$$=\begin{bmatrix}0&0&0&0&1/3&1/3&1/3&1/3&2/3&2/3&2/3&2/3&1&1&1&1\\0&0&1/27&1/9&0&0&1/27&1/9&1/75&1/75&34/675&34/675&1/25&1/25&52/675&34/2\\0&1/3&2/3&1&0&1/3&2/3&1&0&1/3&2/3&1&0&1/3&2/3&1\end{bmatrix} \quad (34)$$

3.2.2 Oriented Hyperbolic Paraboloid

A hyperbolic paraboloid, which should not be mistaken for a hyperboloid, is a doubly ruled surface that takes on a saddle-like shape. In an appropriate 3D coordinate system, its geometry can be described by the equation $z=\frac{x^2}{a^2}-\frac{y^2}{b^2}$. This equation defines a surface that curves upward in the direction of the y-axis and curves downward along the x-axis.

Specifically, if you slice the surface with the plane $x=0$, you'll see a parabola that opens upward, and if you slice it with the plane $y=0$, you'll get a parabola that opens downward.

Theorem 3.2.4. The control points of the oriented hyperbolic paraboloid surfaces $z=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ as bicubic Bézier surface are

$$[P_0\ P_1\ P_2\ P_3\ P_4\ P_5\ P_6\ P_7\ P_8\ P_9\ P_{10}\ P_{11}\ P_{12}\ P_{13}\ P_{14}\ P_{15}]$$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 1/3 & 2/3 & 2/3 & 2/3 & 2/3 & 1 & 1 & 1 & 1 \\ 0 & 1/3 & 2/3 & 1 & 0 & 1/3 & 2/3 & 1 & 0 & 1/3 & 2/3 & 1 & 0 & 1/3 & 2/3 & 1 \\ 0 & 0 & -\frac{1}{3b^2} & -\frac{1}{b^2} & 0 & 0 & -\frac{1}{3b^2} & -\frac{1}{b^2} & \frac{1}{3a^2} & \frac{1}{3a^2} & \frac{1}{3a^2}-\frac{1}{3b^2} & \frac{1}{3a^2}-\frac{1}{b^2} & \frac{1}{a^2} & \frac{1}{a^2} & \frac{1}{a^2}-\frac{1}{3b^2} & \frac{1}{a^2}-\frac{1}{b^2} \end{bmatrix}. \quad (35)$$

Proof. We can write the oriented hyperbolic paraboloid surface as in the parametric form $\varphi(u,v) = \left(u, v, \frac{u^2}{a^2} - \frac{v^2}{b^2}\right)$ with the matrix representation

$$\varphi(u,v) = \left(\begin{bmatrix} v^3 \\ v^2 \\ v \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u^3 \\ u^2 \\ u \\ 1 \end{bmatrix}, \begin{bmatrix} v^3 \\ v^2 \\ v \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u^3 \\ u^2 \\ u \\ 1 \end{bmatrix}, \begin{bmatrix} v^3 \\ v^2 \\ v \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{b^2} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{a^2} & 0 & 0 \end{bmatrix} \begin{bmatrix} u^3 \\ u^2 \\ u \\ 1 \end{bmatrix} \right) \quad (36)$$

hence we have

$$[CMPS]_x = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad [CMPS]_y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad [CMPS]_z = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{b^2} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{a^2} & 0 & 0 \end{bmatrix}. \quad (37)$$

The components of the control points for the any parametric surfaces as cubic Bézier surface are

$$P_x = [B^3]^{-1}[CMPS]_x[B^3]^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix} \quad (38)$$

$$P_y = [B^3]^{-1}[CMPS]_y[B^3]^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad (39)$$

$$P_z = [B^3]^{-1}[CMPS]_z[B^3]^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{b^2} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{a^2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{3a^2} & \frac{1}{a^2} \\ 0 & 0 & \frac{1}{3a^2} & \frac{1}{a^2} \\ \frac{-1}{3b^2} & \frac{-1}{3b^2} & \frac{b^2-a^2}{3a^2b^2} & \frac{3b^2-a^2}{3a^2b^2} \\ \frac{-1}{b^2} & \frac{-1}{b^2} & \frac{b^2-3a^2}{3a^2b^2} & \frac{b^2-a^2}{a^2b^2} \end{bmatrix}. \quad (40)$$

Example 3.2.4. By replacing $a = 1$ and $b = 1$ in theorem 3.2.4, the control points of the oriented hyperbolic paraboloid surfaces $z = x^2 - y^2$ as cubic Bézier surface are

$$[P_0\ P_1\ P_2\ P_3\ P_4\ P_5\ P_6\ P_7\ P_8\ P_9\ P_{10}\ P_{11}\ P_{12}\ P_{13}\ P_{14}\ P_{15}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 1/3 & 2/3 & 2/3 & 2/3 & 2/3 & 1 & 1 & 1 & 1 \\ 0 & 1/3 & 2/3 & 1 & 0 & 1/3 & 2/3 & 1 & 0 & 1/3 & 2/3 & 1 & 0 & 1/3 & 2/3 & 1 \\ 0 & 0 & -1/3 & -1 & 0 & 0 & -1/3 & -1 & 1/3 & 1/3 & 0 & -2/3 & 1 & 1 & 2/3 & 0 \end{bmatrix}. \quad (41)$$

Example 3.2.5. Lets find the control points of the hyperbolic paraboloid $z = \frac{x^2}{25} - \frac{y^2}{9}$ as bicubic Bézier surface. First we can write as

$$\varphi(u,v) = \left(u, v, \frac{u^2}{25} - \frac{v^2}{9}\right) \quad (42)$$

By replacing $a = 5$ and $b = 3$ in [1] as in the following way, the control points are

$$[P_0\ P_1\ P_2\ P_3\ P_4\ P_5\ P_6\ P_7\ P_8\ P_9\ P_{10}\ P_{11}\ P_{12}\ P_{13}\ P_{14}\ P_{15}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 1/3 & 2/3 & 2/3 & 2/3 & 2/3 & 1 & 1 & 1 & 1 \\ 0 & 1/3 & 2/3 & 1 & 0 & 1/3 & 2/3 & 1 & 0 & 1/3 & 2/3 & 1 & 0 & 1/3 & 2/3 & 1 \\ 0 & 0 & -1/27 & -1/9 & 0 & 0 & -1/27 & -1/9 & 1/75 & 1/75 & -16/675 & -22/225 & 1/25 & 1/25 & 2/675 & -16/2025 \end{bmatrix}. \quad (43)$$

Corollary 3.2.1. The hyperbolic paraboloid $y = x^2 - z^2$ as bicubic Bézier surface with given 16 control points

$$[P_0\ P_1\ P_2\ P_3\ P_4\ P_5\ P_6\ P_7\ P_8\ P_9\ P_{10}\ P_{11}\ P_{12}\ P_{13}\ P_{14}\ P_{15}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 1/3 & 2/3 & 2/3 & 2/3 & 2/3 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1/3 & -1 & 0 & 0 & -1/3 & -1 & 1/3 & 1/3 & 0 & -2/3 & 1 & 1 & 2/3 & 0 \\ 0 & 1/3 & 2/3 & 1 & 0 & 1/3 & 2/3 & 1 & 0 & 1/3 & 2/3 & 1 & 0 & 1/3 & 2/3 & 1 \end{bmatrix}. \quad (44)$$

is an the hyperbolic paraboloid $y = x^2 - z^2$ surface, if the following equation is taken into account

$$\varphi(u,v) = [v]^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_{00} & P_{10} & P_{20} & P_{30} \\ P_{01} & P_{11} & P_{21} & P_{31} \\ P_{02} & P_{12} & P_{22} & P_{32} \\ P_{03} & P_{13} & P_{23} & P_{33} \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} [u]^T \quad (45)$$

$$= \begin{bmatrix} v^3 \\ v^2 \\ v \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u^3 \\ u^2 \\ u \\ 1 \end{bmatrix} \quad (46)$$

$$\begin{bmatrix} v^3 \\ v^2 \\ v \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{1}{3} & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ -\frac{1}{3} & -\frac{1}{3} & 0 & \frac{2}{3} \\ -1 & -1 & -\frac{2}{3} & 0 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u^3 \\ u^2 \\ u \\ 1 \end{bmatrix} \quad (47)$$

$$\begin{bmatrix} v^3 \\ v^2 \\ v \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u^3 \\ u^2 \\ u \\ 1 \end{bmatrix} \tag{48}$$

$$\varphi(u,v) = (u, \quad u^2 - v^2, \quad v). \tag{49}$$

3.2.3 Parabolic Cylinder

Parabolic cylinder can be represented by the equation $z = ax^2$.

Theorem 3.2.5. The control points of the oriented parabolic cylinder surfaces $z = ax^2$ as bicubic Bézier surface are

$$\begin{aligned} &[P_0 \quad P_1 \quad P_2 \quad P_3 \quad P_4 \quad P_5 \quad P_6 \quad P_7 \quad P_8 \quad P_9 \quad P_{10} \quad P_{11} \quad P_{12} \quad P_{13} \quad P_{14} \quad P_{15}] \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 1 & 1 & 1 & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 & 0 & \frac{1}{3} & \frac{2}{3} & 1 & 0 & \frac{1}{3} & \frac{2}{3} & 1 & 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3a} & \frac{1}{3a} & \frac{1}{3a} & \frac{1}{3a} & a & a & a & a \end{bmatrix}. \end{aligned} \tag{50}$$

Proof. We can write the oriented parabolic cylinder surfaces $\varphi(u,v) = (u, v, au^2)$ with the matrix representation

$$\varphi(u,v) = \left(\begin{bmatrix} v^3 \\ v^2 \\ v \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u^3 \\ u^2 \\ u \\ 1 \end{bmatrix}, \begin{bmatrix} v^3 \\ v^2 \\ v \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u^3 \\ u^2 \\ u \\ 1 \end{bmatrix}, \begin{bmatrix} v^3 \\ v^2 \\ v \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{bmatrix} \begin{bmatrix} u^3 \\ u^2 \\ u \\ 1 \end{bmatrix} \right). \tag{51}$$

The components of the control points for any parametric surfaces as cubic Bézier surface are

$$[P_x] = [B^3]^{-1}[CMPS]_x[B^3]^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix} \tag{52}$$

$$[P_y] = [B^3]^{-1}[CMPS]_y[B^3]^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{53}$$

$$[P_z] = [B^3]^{-1}[CMPS]_z[B^3]^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{3}a & a \\ 0 & 0 & \frac{1}{3}a & a \\ 0 & 0 & \frac{1}{3}a & a \\ 0 & 0 & \frac{1}{3}a & a \end{bmatrix} \tag{54}$$

Corollary 3.2.2. The control points of the oriented parabolic cylinder surfaces $y = ax^2$ or $\varphi(u,v) = (u, au^2, v)$ as bicubic Bézier surface are

$$[P_0 \;\; P_1 \;\; P_2 \;\; P_3 \;\; P_4 \;\; P_5 \;\; P_6 \;\; P_7 \;\; P_8 \;\; P_9 \;\; P_{10} \;\; P_{11} \;\; P_{12} \;\; P_{13} \;\; P_{14} \;\; P_{15}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 1/3 & 2/3 & 2/3 & 2/3 & 2/3 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/3a & 1/3a & 1/3a & 1/3a & a & a & a & a \\ 0 & 1/3 & 2/3 & 1 & 0 & 1/3 & 2/3 & 1 & 0 & 1/3 & 2/3 & 1 & 0 & 1/3 & 2/3 & 1 \end{bmatrix}, \tag{55}$$

if the following equation taken into account

$$\varphi(u,v) = \left(\begin{bmatrix} v^3 \\ v^2 \\ v \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u^3 \\ u^2 \\ u \\ 1 \end{bmatrix}, \begin{bmatrix} v^3 \\ v^2 \\ v \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{bmatrix} \begin{bmatrix} u^3 \\ u^2 \\ u \\ 1 \end{bmatrix}, \begin{bmatrix} v^3 \\ v^2 \\ v \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u^3 \\ u^2 \\ u \\ 1 \end{bmatrix} \right) \tag{56}$$

Example 3.2.6. Lets find the control points of parabolic cylinder $z = 4x^2$ as bicubic Bézier surface. First we can write as $\varphi(u,v) = (u, v, 4u^2)$. For the value of $a = 4$, the control points are

$$[P_0 \;\; P_1 \;\; P_2 \;\; P_3 \;\; P_4 \;\; P_5 \;\; P_6 \;\; P_7 \;\; P_8 \;\; P_9 \;\; P_{10} \;\; P_{11} \;\; P_{12} \;\; P_{13} \;\; P_{14} \;\; P_{15}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 1/3 & 2/3 & 2/3 & 2/3 & 2/3 & 1 & 1 & 1 & 1 \\ 0 & 1/3 & 2/3 & 1 & 0 & 1/3 & 2/3 & 1 & 0 & 1/3 & 2/3 & 1 & 0 & 1/3 & 2/3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/12 & 1/12 & 1/12 & 1/12 & 4 & 4 & 4 & 4 \end{bmatrix}. \tag{57}$$

4 CONCLUSION AND SUGGESTIONS

Bicubic Bézier surfaces enable mathematical description and control of complex and smooth surfaces. This feature makes them indispensable in many fields, from engineering to art. Bicubic Bézier surfaces are particularly important in the following areas:

- Used in 3D modeling and animation (for example by studios like Pixar) to smoothly render complex surfaces and characters. Allows precise control of object surfaces, which is critical for realistic visuals.
- It is used in automotive, aviation and industrial design, modeling of vehicle bodies and product surfaces. It is very valuable in terms of engineering since the curvature and transitions of the surface can be controlled precisely.
- It provides representation of models with smooth surfaces in the production of parts requiring precise geometry. Accurate surface representation is important in pre-production simulations and analyses.
- Used to describe complex geometries in areas such as fluid mechanics, biomedical modeling. For example: complex structures such as organ surfaces or airflow surfaces.

In this study, some special surface types such as bicubic Bézier surface were investigated and practical and useful methods were given for the analysis of these surfaces. Authors from different disciplines and fields of study can also find the opportunity to easily analyze the surface types they need to use by using the methods in the paper.

Conflict of Interest Statement

There is no conflict of interest between the authors.

Statement of Research and Publication Ethics

The study is complied with research and publication ethics.

Artificial Intelligence (AI) Contribution Statement

This manuscript was entirely written, edited, analyzed, and prepared without the assistance of any artificial intelligence (AI) tools. All content, including text, data analysis, and figures, was solely generated by the authors.

Contributions of the Authors

Ş.K.: Conceptualization, Methodology, Computation, and Formal Analysis, S.Y.: Writing, Validation, Reviewing, Editing, Original Draft Preparation, and Investigation.

REFERENCES

[1] S. Michael, "Bézier Curves and Surfaces", Lecture 8, Floater Oslo Oct., 2003.

[2] H. Hagen, "Bézier Curves with Curvature and Torsion Continuity", *The Rocky Mountain Journal of Mathematics*, Vol. 16, no. 3, pp. 629-638, 1986.

[3] G. Farin, "Designing C1 Surfaces Consisting of Triangular Cubic Patches", *Computer-Aided Design*, vol. 14, no. 5, pp. 253-256, 1982.

[4] S. N. Krivoshapko, and V. N. Ivanov, "Surfaces of Bézier", *Encyclopedia of Analytical Surfaces*, pp. 327-329, 2015.

[5] S. N. I. Rosli, and M. I. E. Zulkifly, "Neutrosophic Bicubic Bézier Surface Approximation Model for Uncertainty Data", *Matematika*, vol. 39, pp. 281-291, 2023.

[6] S. Bashir, D. Ahmad, and G. Ali, "Exploring q-Bernstein-Bézier Surfaces in Minkowski Space: Analysis, Modeling, and Applications", *Plos one*, vol. 19, no. 5, pp. 1-26, 2024.

[7] Ş. Kılıçoğlu, "On Approximation of Helix 3rd, 5th and 7th Order Bézier Curves in E^3", *Thermal Science*, vol. 26, Special Issue 2, pp. 525-538, 2022.

[8] Ş. Kılıçoğlu, "On Approximation Sine Wave with the 5th and 7th Order Bézier Paths in Plane", *Thermal Science*, vol. 26, Special Issue 2, pp. 539-550, 2022.

[9] Ş. Kılıçoğlu, and S. Yurttancikmaz, "How to Approximate Cosine Curve with 4th and 6th Order Bézier Curve in Plane?", *Thermal Science*, vol. 26, Special Issue 2, pp. 559-570, 2022.

[10] Ş. Kılıçoğlu, and S. Yurttancikmaz, "A Modelling of the Natural Logarithm and Mercator Series as 5^th, 6^th, 7^th Order Bézier Curve in Plane", *MANAS Journal of Engineering*, vol. 12, no. 2, pp. 185-191, 2024.

[11] Ş. Kılıçoğlu, and S. Yurttancikmaz, "A Modelling on the Exponential Curves as Cubic, 5th and 7th Bézier Curve in Plane", *CAMS*, vol. 6, no. 2, pp. 67-77, 2023.

[12] Ş. Kılıçoğlu, and S. Yurttancikmaz, "On The Some Special Biquadratic Bézier Surfaces in E^3", *Journal of the Institute of Science and Technology*, vol. 15, no. 3, pp. 1110-1123, 2025.

[13] Ş. Kılıçoğlu, and S. Şenyurt, "On the Bertrand Mate of Cubic Bézier Curve by Using Matrix Representation in E³", *Turkish Journal of Mathematics and Computer Science*, vol. 14, no. 2, pp. 376-383, 2021.

[14] Ş. Kılıçoğlu, and S. Şenyurt, "On the Cubic Bézier Curves in E³", *Ordu University Journal of Science and Technology*, vol. 9, no. 2, pp. 83-97, 2019.

[15] Ş. Kılıçoğlu, and S. Şenyurt, "On the Involute of the Cubic Bézier Curve by Using Matrix Representation in E³", *European Journal of Pure and Applied Mathematics*, vol. 13, no. 2, pp. 216-226, 2020.

[16] Ş. Kılıçoğlu, and S. Şenyurt, "On the Matrix Representation of 5th Order Bézier Curve and Derivatives in E³", *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, vol. 71, no. 1, pp. 133-152, 2022.

[17] G. Farin, *Curves and Surfaces for CAGD (5th ed.)*. Academic Press. ISBN 1-55860-737-4, (2002).

[18] J. Zheng, and T.W. Sederberg, "Gaussian and Mean Curvature of Rational Bézier Patches", *Computer Aided Geometric Design*, vol. 20, no. 6, pp. 297-301, 2003.

[19] Practical Guide to Bézier Surfaces, [Online]. Available: https://www.gamedev.net/tutorials/programming/math-and-physics/practical-guide-to-bezier-surfaces-r3170/

[20] Numerical Computation Lecture 14: Bézier Curves United International College, [Online]. Available: https://slideplayer.com/slide/7592222/

[21] T.W. Sederberg, D.C. Anderson, and R.N. Goldman, "Implicit Representation of Parametric Curves and Surfaces", *Computer Vision, Graphics, and Image Processing*, vol. 28, no. 1, pp. 72-84, 1984.

[22] M. Ammad, M.Y. Misro, M. Abbas, and A. Majeed, "Generalized Developable Cubic Trigonometric Bézier Surfaces", *Mathematics*, vol. 9, no. 3, 283, 2021.

[23] M. Geimer, O. Abert, "Interactive Ray Tracing of Trimmed Bicubic Bézier Surfaces Without Triangulation", *Proceedings of WSCG*, pp. 71-78, 2005.

[24] X. A. Han, Y. Ma, and X. Huang, "The Cubic Trigonometric Bézier Curve with Two Shape Parameters", *Applied Mathematics Letters*, vol. 22, no. 2, pp. 226-231, 2009.

[25] S.N.I. Rosli, and M.I.E. Zulkifly, "Neutrosophic Bicubic Bezier Surface Approximationmodel for Uncertainty Data", *Matematika*, pp. 281-291, 2023.

[26] M. Ameer, M. Abbas, T. Abdeljawad, and T. Nazir, "A Novel Generalization of Bézier-like Curves and Surfaces with Shape Parameters", *Mathematics*, vol. 10, no. 3, 376, 2022.